Metatheory of the Logic of Hereditary Harrop Formulas in Coq

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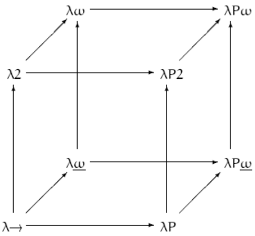
Coq

Interactive theorem prover developed at

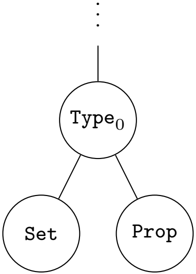
Inria

Implementation of extension of Calculus of Constructions (CoC) created by Thierry Coquand



← CoC

Calculus of Constructions



Notation: $A : B$ means A has type B

If $P : \texttt{Prop}$ and $t : P$, then P is a theorem and t is a proof of P

Example:

$$(\forall (n : \texttt{nat}), 0 + n = n) : \texttt{Prop}$$
$$(\texttt{fun}\ (n : \texttt{nat}) \Rightarrow \texttt{eq\_refl}) : \forall (n : \texttt{nat}), 0 + n = n$$

Inference Rule

$$\frac{P_1 \quad \cdots \quad P_n}{C} \; name$$

Vertical "implication" notation

Inference Rule

$$\frac{P_1 \quad \cdots \quad P_n}{C}\ name$$

Vertical “implication” notation

- If P_1, ..., P_n, then C

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Vertical "implication" notation

- If P_1, ..., P_n, then C
- To prove C, show P_1, ..., P_n all true

Inference Rule

$$\frac{P_1 \quad \cdots \quad P_n}{C}\ name$$

Vertical "implication" notation

- If P_1, ..., P_n, then C
- To prove C, show P_1, ..., P_n all true
- If you can build P_1, ..., P_n, then you can build C

Natural Deduction

Set of inference rules to encode "natural" reasoning

e.g. $\dfrac{A \quad B}{A \wedge B} \wedge_I \qquad \dfrac{A \wedge B}{A} \wedge_{E_1} \qquad \dfrac{A \wedge B}{B} \wedge_{E_2}$

Claim: If $p \wedge q$, then $q \wedge p$

Proof:

$$\dfrac{\dfrac{p \wedge q}{q} \wedge_{E_2} \qquad \dfrac{p \wedge q}{p} \wedge_{E_1}}{q \wedge p} \wedge_I$$

Sequent Calculus

Sequent

$$\Gamma \vdash P$$

P is provable in context Γ (a set of assumptions)

Example Sequent Rule

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \to_I$$

Prove "If P then Q" by assuming P and deriving Q

Sequents as Dependent Types in Coq

Goal-Reduction Sequent

$\texttt{grseq} : \texttt{context} \rightarrow \texttt{oo} \rightarrow \texttt{Prop}$

$\Gamma \rhd \beta$ is notation for $\texttt{grseq}\ \Gamma\ \beta$

Backchaining Sequent

$\texttt{bcseq} : \texttt{context} \rightarrow \texttt{oo} \rightarrow \texttt{atm} \rightarrow \texttt{Prop}$

$\Gamma, [\beta] \rhd \alpha$ is notation for $\texttt{bcseq}\ \Gamma\ \beta\ \alpha$

[Wang, Chaudhuri, Gacek, Nadathur; 2012]

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[Wang, Chaudhuri, Gacek, Nadathur; 2012]

The Specification Logic (Hereditary Harrop)

Goal-Reduction Rules

$$\frac{A :- G \quad \Gamma \rhd G}{\Gamma \rhd \langle\, A \,\rangle}\ \text{g\_prog} \qquad \frac{D \in \Gamma \quad \Gamma, [D] \rhd A}{\Gamma \rhd \langle\, A \,\rangle}\ \text{g\_dyn} \qquad \frac{\Gamma \rhd G_1 \quad \Gamma \rhd G_2}{\Gamma \rhd G_1 \,\&\, G_2}\ \text{g\_and}$$

$$\frac{\Gamma\,,\, D \rhd G}{\Gamma \rhd D \longrightarrow G}\ \text{g\_imp} \qquad \frac{}{\Gamma \rhd \mathtt{T}}\ \text{g\_tt} \qquad \frac{\mathtt{proper}\ E \quad \Gamma \rhd G\ E}{\Gamma \rhd \mathtt{Some}\ G}\ \text{g\_some}$$

$$\frac{\forall (E : \mathtt{expr\ \ con}), (\mathtt{proper}\ E \to \Gamma \rhd G\ E)}{\Gamma \rhd \mathtt{All}\ G}\ \text{g\_all} \qquad \frac{\forall (E : \mathtt{X}), (\Gamma \rhd G\ E)}{\Gamma \rhd \mathtt{Allx}\ G}\ \text{g\_allx}$$

Backchaining Rules

$$\frac{}{\Gamma, [\langle\, A \,\rangle] \rhd A}\ \text{b\_match} \qquad \frac{\Gamma, [D_1] \rhd A}{\Gamma, [D_1 \,\&\, D_2] \rhd A}\ \text{b\_and1} \qquad \frac{\Gamma, [D_2] \rhd A}{\Gamma, [D_1 \,\&\, D_2] \rhd A}\ \text{b\_and2}$$

$$\frac{\Gamma \rhd G \quad \Gamma, [D] \rhd A}{\Gamma, [G \longrightarrow D] \rhd A}\ \text{b\_imp} \qquad \frac{\mathtt{proper}\ E \quad \Gamma, [D\ E] \rhd A}{\Gamma, [\mathtt{All}\ D] \rhd A}\ \text{b\_all} \qquad \frac{\Gamma, [D\ E] \rhd A}{\Gamma, [\mathtt{Allx}\ D] \rhd A}\ \text{b\_allx}$$

$$\frac{\forall (E : \mathtt{expr\ \ con}), (\mathtt{proper}\ E \to \Gamma, [D\ E] \rhd A)}{\Gamma, [\mathtt{Some}\ D] \rhd A}\ \text{b\_some}$$

The Specification Logic (Hereditary Harrop)

Goal-Reduction Rules

$$\frac{A \mathrel{:-} G \quad \Gamma \rhd G}{\Gamma \rhd \langle\, A \,\rangle}\ \text{g\_prog} \qquad \frac{D \in \Gamma \quad \Gamma, [D] \rhd A}{\Gamma \rhd \langle\, A \,\rangle}\ \text{g\_dyn} \qquad \boxed{\frac{\Gamma \rhd G_1 \quad \Gamma \rhd G_2}{\Gamma \rhd G_1 \,\&\, G_2}\ \text{g\_and}}$$

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$$\frac{\forall (E : \mathtt{expr\ \ con}), (\mathtt{proper}\ E \to \Gamma, [D\ E] \rhd A)}{\Gamma, [\mathtt{Some}\ D] \rhd A}\ \text{b\_some}$$

Encoding Sequents as Inductive Dependent Types

$$\vdots$$

$$\frac{\Gamma \rhd G_1 \quad \Gamma \rhd G_2}{\Gamma \rhd G_1 \,\&\, G_2}\ \text{g\_and}$$

$$\vdots$$

$$\frac{\Gamma \rhd G \quad \Gamma, [D] \rhd A}{\Gamma, [G \longrightarrow D] \rhd A}\ \text{b\_imp}$$

$$\vdots$$

```
Inductive grseq : context -> oo -> Prop :=
...
| g_and :
 forall (L : context) (G1 G2 : oo),
 grseq L G1 -> grseq L G2 ->
 grseq L (G1 & G2)
...
with bcseq : context -> oo ->  atm -> Prop :=
| b_imp :
 forall (L : context) (F G : oo) (A : atm),
 grseq L G -> bcseq L D A
  -> bcseq L (G ---> D) A.
...
```

Inductive Types

Encapsulate infinite collection in finite set of rules

Example:

```
Inductive nat : Set :=
| O : nat
| S : nat -> nat.
```

Induction principle for property $P : \texttt{nat} \to \texttt{Prop}$:

$$\frac{\mathrm{P}\ \ 0 \quad \forall(\mathrm{n} : \mathrm{nat}), \mathrm{P}\ \ \mathrm{n} \to \mathrm{P}\ \ (\mathrm{S}\ \ \mathrm{n})}{\forall(\mathrm{n} : \mathrm{nat}), \mathrm{P}\ \ \mathrm{n}}$$

In linear form:

$$\begin{aligned}
&\forall(P : \texttt{nat} \to \texttt{Prop}), \\
&\quad (P\ 0) \to \\
&\quad (\forall(n : \texttt{nat}), P\ n \to P\ (S\ n)) \to \\
&\ \forall(n : \texttt{nat}), P\ n.
\end{aligned}$$

Sequent Mutual Induction Principle

$$\vdots$$

$$\frac{\Gamma \rhd G_1 \quad \Gamma \rhd G_2}{\Gamma \rhd G_1 \,\&\, G_2}\ \text{g\_and}$$

$$\vdots$$

$$\frac{\Gamma \rhd G \quad \Gamma, [D] \rhd A}{\Gamma, [G \longrightarrow D] \rhd A}\ \text{b\_imp}$$

$$\vdots$$

$$
\begin{aligned}
\texttt{seq\_mutind} : \forall (P_1 &: \texttt{context} \to \texttt{oo} \to \texttt{Prop}) \\
&(P_2 : \texttt{context} \to \texttt{oo} \to \texttt{atm} \to \texttt{Prop}), \\
&\vdots \\
&(\forall (c : \texttt{context})(o_1\ o_2 : \texttt{oo}), \\
&\quad c \rhd o_1 \to P_1\ c\ o_1 \to c \rhd o_2 \to P_1\ c\ o_2 \to \\
&\quad P_1\ c\ (o_1 \& o_2)) \to \\
&\vdots \\
&(\forall (c : \texttt{context})(o_1\ o_2 : \texttt{oo})(a : \texttt{atm}), \\
&\quad c \rhd o_1 \to P_1\ c\ o_1 \to c, [o_2] \rhd a \to P_2\ c\ o_2\ a \to \\
&\quad P_2\ c\ (o_1 \longrightarrow o_2)\ a) \to \\
&\vdots \\
(\forall (c :\ &\texttt{context})(o : \texttt{oo}), \\
&\qquad c \rhd o \to P_1\ c\ o) \wedge \\
(\forall (c :\ &\texttt{context})(o : \texttt{oo})(a : \texttt{atm}), \\
&\qquad c, [o] \rhd a \to P_2\ c\ o\ a)
\end{aligned}
$$

Structural Rules

$$\frac{\Gamma \rhd \beta_2}{\Gamma\,,\beta_1 \rhd \beta_2}\ \texttt{gr\_weakening} \qquad \frac{\Gamma, [\beta_2] \rhd \alpha}{\Gamma\,,\beta_1, [\beta_2] \rhd \alpha}\ \texttt{bc\_weakening}$$

$$\frac{\Gamma\,,\ \beta_1\,,\beta_1 \rhd \beta_2}{\Gamma\,,\beta_1 \rhd \beta_2}\ \texttt{gr\_contraction} \qquad \frac{\Gamma\,,\ \beta_1\,,\ \beta_1, [\beta_2] \rhd \alpha}{\Gamma\,,\ \beta_1, [\beta_2] \rhd \alpha}\ \texttt{bc\_contraction}$$

$$\frac{\Gamma\,,\ \beta_2\,,\ \beta_1 \rhd \beta_3}{\Gamma\,,\beta_1\,,\ \beta_2 \rhd \beta_3}\ \texttt{gr\_exchange} \qquad \frac{\Gamma\,,\ \beta_2\,,\ \beta_1, [\beta_3] \rhd \alpha}{\Gamma\,,\ \beta_1\,,\ \beta_2, [\beta_3] \rhd \alpha}\ \texttt{bc\_exchange}$$

These are all corollaries of a general theorem:

Theorem (`monotone`)

$$\frac{\Gamma \subseteq \Gamma' \quad \Gamma \rhd \beta}{\Gamma' \rhd \beta} \wedge \frac{\Gamma \subseteq \Gamma' \quad \Gamma, [\beta] \rhd \alpha}{\Gamma', [\beta] \rhd \alpha}$$

Theorem (monotone)

$$
\begin{aligned}
&(\ \forall(\Gamma : \texttt{context})(\beta : \texttt{oo}),\\
&\qquad \Gamma \rhd \beta \to \forall(\Gamma' : \texttt{context}), \Gamma \subseteq \Gamma' \to \Gamma' \rhd \beta\ \ ) \wedge\\
&(\ \forall(\Gamma : \texttt{context})(\beta : \texttt{oo})(\alpha : \texttt{atm}),\\
&\qquad \Gamma, [\beta] \rhd \alpha \to \forall(\Gamma' : \texttt{context}), \Gamma \subseteq \Gamma' \to \Gamma', [\beta] \rhd \alpha\ \ )
\end{aligned}
$$

Define

$$
\begin{aligned}
P_1 &:= \lambda(\Gamma : \texttt{context})(\beta : \texttt{oo})\ .\\
&\qquad \forall(\Gamma' : \texttt{context}), \Gamma \subseteq \Gamma' \to \Gamma' \rhd \beta\\
P_2 &:= \lambda(\Gamma : \texttt{context})(\beta : \texttt{oo})(\alpha : \texttt{atm})\ .\\
&\qquad \forall(\Gamma' : \texttt{context}), \Gamma \subseteq \Gamma' \to \Gamma', [\beta] \rhd \alpha
\end{aligned}
$$

Proof

By induction over $\Gamma \rhd \beta$ and $\Gamma, [\beta] \rhd \alpha$ using `seq_mutind`.

Theorem (monotone)

$$
\begin{aligned}
&(\ \forall(\Gamma : \texttt{context})(\beta : \texttt{oo}),\\
&\qquad \Gamma \rhd \beta \rightarrow \boxed{\forall(\Gamma' : \texttt{context}), \Gamma \subseteq \Gamma' \rightarrow \Gamma' \rhd \beta}\ ) \land\\
&(\ \forall(\Gamma : \texttt{context})(\beta : \texttt{oo})(\alpha : \texttt{atm}),\\
&\qquad \Gamma, [\beta] \rhd \alpha \rightarrow \forall(\Gamma' : \texttt{context}), \Gamma \subseteq \Gamma' \rightarrow \Gamma', [\beta] \rhd \alpha\ \ )
\end{aligned}
$$

Define

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P_1 &\coloneqq \lambda(\Gamma : \texttt{context})(\beta : \texttt{oo})\ .\\
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\end{aligned}
$$

Proof

By induction over $\Gamma \rhd \beta$ and $\Gamma, [\beta] \rhd \alpha$ using `seq_mutind`.

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&(\ \forall(\Gamma : \mathtt{context})(\beta : \mathtt{oo})(\alpha : \mathtt{atm}),\\
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&\qquad \boxed{\forall(\Gamma' : \mathtt{context}), \Gamma \subseteq \Gamma' \to \Gamma', [\beta] \rhd \alpha}
\end{aligned}
$$

Proof

By induction over $\Gamma \rhd \beta$ and $\Gamma, [\beta] \rhd \alpha$ using `seq_mutind`.

Proof of `monotone`

Case $\dfrac{\Gamma\ ,\ D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

$$\dfrac{\begin{array}{l} H\ : c, o_1 \rhd o_2 \\ IH\ : P_1\ (c, o_1)\ o_2 \end{array}}{P_1\ c\ (o_1 \longrightarrow o_2)}$$

Proof of `monotone`

Case $\dfrac{\Gamma\,,\; D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

$$\dfrac{\begin{array}{l} H \;:\, c, o_1 \rhd o_2 \\ IH \;:\, P_1 \; (c, o_1) \; o_2 \end{array}}{P_1 \; c \; (o_1 \longrightarrow o_2)}$$

Next: unfold P_1

Proof of `monotone`

Case $\dfrac{\Gamma\,,\ D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

$$\dfrac{\begin{array}{l} H\ : c, o_1 \rhd o_2 \\ IH\ : \forall(\Gamma_0 : \texttt{context}), (c, o_1) \subseteq \Gamma_0 \to \Gamma_0 \rhd o_2 \end{array}}{\forall(\Gamma' : \texttt{context}), c \subseteq \Gamma' \to \Gamma' \rhd o_1 \longrightarrow o_2}$$

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Next: introduce from goal

Proof of `monotone`

Case $\dfrac{\Gamma\,,\; D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

$$\begin{array}{r l} H & : c, o_1 \rhd o_2 \\ IH & : \forall (\Gamma_0 : \texttt{context}), (c, o_1) \subseteq \Gamma_0 \to \Gamma_0 \rhd o_2 \\ P & : c \subseteq \Gamma' \\ \hline & \Gamma' \rhd o_1 \longrightarrow o_2 \end{array}$$

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Next: backchain with g_imp on goal

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Case $\dfrac{\Gamma\,,\ D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

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Next: backchain with IH on goal with $(\Gamma_0 := \Gamma', o_1)$

Proof of `monotone`

Case $\dfrac{\Gamma\,,\; D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

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Proof of `monotone`

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Next: backchain with context lemma that says if $c \subseteq \Gamma'$ then $(c, o_1) \subseteq (\Gamma', o_1)$

Proof of `monotone`

Case $\dfrac{\Gamma\,,\;D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

From `seq_mutind`, proving

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Case $\dfrac{\Gamma\,,\ D \rhd G}{\Gamma \rhd D \longrightarrow G}$ **g_imp:**

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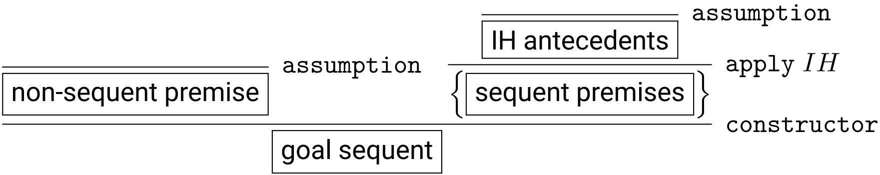
Goal is provable by assumption P

Structural Rules Summary

Proof with 15 subcases proven automatically in Coq




```
Proof.
Hint Resolve context_sub_sup.
eapply seq_mutind; intros;
try (econstructor; eauto; eassumption).
Qed.
```

Prove cut admissibility with this one weird trick...

Theorem (`cut_admissible`)

$$\frac{\Gamma, \delta \rhd \beta \quad \Gamma \rhd \delta}{\Gamma \rhd \beta} \wedge \frac{\Gamma, \delta, [\beta] \rhd \alpha \quad \Gamma \rhd \delta}{\Gamma, [\beta] \rhd \alpha}$$

Proof by nested induction over δ then mutual structural induction over $\Gamma, \delta \rhd \beta$ and $\Gamma, \delta, [\beta] \rhd \alpha$

98 of 105 cases proven automatically in Coq

```
Proof.
Hint Resolve gr_weakening context_swap.
induction delta; eapply seq_mutind; intros;
subst; try (econstructor; eauto; eassumption).
...
```

[Battell, Felty; LFMTP-16]

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- Coq type system is useful for formal theorem proving

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- Coq type system is useful for formal theorem proving
- Rich type systems make induction even more awesome
- Inference systems can be encoded as inductive types in Coq
- Structural properties of logics can be proven in Coq

Thank you!